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# Stability analysis and oscillatory structures in time-fractional reaction-diffusion systems

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The linear stage of stability is studied for a two-component fractional reaction-diffusion system. It is shown that, with a certain value of the fractional derivative index, a different type of instability occurs. The linear stability analysis shows that the system becomes unstable toward perturbations of finite wave number. As a result, inhomogeneous oscillations with this wave number become unstable and lead to nonlinear oscillations which result in spatial oscillatory structure formation. A computer simulation of a Bonhoeffer–van der Pol type of reaction-diffusion system with fractional time derivatives is performed.

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## I. INTRODUCTION

Since the well-known Turing reaction-diffusion model was discovered [1], enormous efforts have been made in the investigation of nonlinear self-organization phenomena in nature. The key models that made the greatest impact, not only in physics, chemistry, or biology, but also in nonlinear science, are the phenomenological models like the Brusselator and Oregonator, the model of Gierer and Meinhardt, the Bonhoeffer–van der Pol model, etc. (to review the models, see, for example, [2–4]). These models make it possible to grasp the general properties of real systems, which at the same time are so complex that they cannot be described just by simple equations. For instance, recent investigations show that, in living systems, the morphogen diffusion in cell environment is non-Fickian and it certainly does not satisfy the standard reaction-diffusion system model [5]. Anomalous diffusion is also inherent to certain plasma systems [6], heterogeneous solid state materials, etc. [7,8]. Therefore, in recent years, there has been a great deal of interest in fractional reaction-diffusion systems (RDSs) [9–17].

In this Rapid Communication, we will show that in fractional reaction-diffusion systems we have a type of instability that is not possible to find in reaction-diffusion systems with integer derivatives. We confirm the linear stability analysis by numerical calculation of a Bonhoeffer–van der Pol type of fractional RDS.

Let us consider the RDS for activator $n_1$ and inhibitor $n_2$

$$\tau_1 \frac{\partial^\alpha n_1(x,t)}{\partial t^\alpha} = l^2 \frac{\partial^2}{\partial x^2} n_1(x,t) + W(n_1,n_2,\mathcal{A}), \tag{1}$$

$$\tau_2 \frac{\partial^\alpha n_2(x,t)}{\partial t^\alpha} = L^2 \frac{\partial^2}{\partial x^2} n_2(x,t) + Q(n_1,n_2,\mathcal{A}) \tag{2}$$

subject to the Neumann boundary conditions

$$dn_i/dx|_{x=0} = dn_i/dx|_{x=l_x} = 0, \quad i = 1,2, \tag{3}$$

and with certain initial condition $n_i|_{t=0}=n_i^0(x)$. Here $x$, $0 \leqslant x \leqslant l_x$, $\tau_1$, $\tau_2$, $l$, and $L$ are the characteristic times and lengths of the system, and $\mathcal{A}$ is an external parameter.

The fractional derivatives $\partial^\alpha n_i(x,t)/\partial t^\alpha$ on the left-hand side of Eqs. (1) and (2), instead of the standard time derivatives, are the Caputo fractional derivatives in time of order $0<\alpha<2$ and are represented by [18,19]

$$\frac{\partial^\alpha n_i(t)}{\partial t^\alpha} := \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{n_i^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau,$$

where $m-1<\alpha<m$, $m \in \overline{1,2}$ (see, [18,19]).

## II. LINEAR STABILITY ANALYSIS

The stability of the steady-state constant solutions of the system (1) and (2) corresponding to the homogeneous equilibrium state

$$W(n_1,n_2,\mathcal{A}) = 0, \quad Q(n_1,n_2,\mathcal{A}) = 0 \tag{4}$$

can be analyzed by linearization of the system near this solution. In this case, the system (1) and (2) can be transformed to a linear fractional RDS at this equilibrium point and eventually can be converted to the simplest diagonal representation

$$\frac{d^\alpha \Delta n_i(t)}{dt^\alpha} = \lambda_i \Delta n_i(t), \tag{5}$$

where $\lambda_{1,2}=\frac{1}{2}(\mathrm{tr}F \pm \sqrt{\mathrm{tr}^2 F - 4\det F})$ are eigenvalues of the matrix

$$F = \begin{pmatrix} (a_{11}-k^2l^2)/\tau_1 & a_{12}/\tau_1 \\ a_{21}/\tau_2 & (a_{22}-k^2L^2)/\tau_2 \end{pmatrix},$$

$k=(\pi/l_x)j$, $j=1,2,\ldots$, $a_{11}=W'_{n_1}$, $a_{12}=W'_{n_2}$, $a_{21}=Q'_{n_1}$, and $a_{22}=Q'_{n_2}$ (all derivatives are taken at homogeneous equilibrium states $W=Q=0$), and $\Delta n_i$ are new variables obtained as a result of the change of basis corresponding to diagonalization of the matrix $F$.

In this case, the solution of the vector equation (5) is given by the Mittag-Leffler functions [18,19]

 

$$\Delta n_i(t) = \sum_{k=0}^{\infty} \frac{(\lambda_i t^{\alpha})^k \Delta n_i(0)}{\Gamma(k\alpha+1)} = E_{\alpha}(\lambda_i t^{\alpha}) \Delta n_i(0), \quad i=1,2. \tag{6}$$

Analyzing (6), we can conclude that, if for any of the roots

$$|\arg(\lambda_i)| < \alpha\pi/2 \tag{7}$$

the solution has an increasing function component, then the system is asymptotically unstable [20].

For integer $\alpha=1$ the roots $\lambda_{1,2}$ are complex inside the parabola $\det F = \mathrm{tr}^2 F/4$, and the fixed points are spiral sources ($\mathrm{tr}F>0$) or spiral sinks ($\mathrm{tr}F<0$). In this case, the domain on the right-hand side of the parabola ($\mathrm{tr}F>0$) is unstable with the existing limit cycle, while the domain on the left-hand side ($\mathrm{tr}F<0$) is stable. By crossing the axis $\mathrm{tr}F=0$, the Hopf bifurcation conditions become true.

For $\alpha$, $0<\alpha<2$, for every point inside the parabola $\det F = \mathrm{tr}^2 F/4$, we can introduce a marginal value $\alpha_0$, $\alpha_0 = (2/\pi)|\arg(\lambda_i)|$, which follows from the equality conditions (7) and is given by the formula [15]

$$\alpha_0 = \begin{cases} \dfrac{2}{\pi} \arctan \sqrt{4 \det F/\mathrm{tr}^2 F - 1}, & \mathrm{tr}F \geqslant 0, \\ 2 - \dfrac{2}{\pi} \arctan \sqrt{4 \det F/\mathrm{tr}^2 F - 1}, & \mathrm{tr}F \leqslant 0. \end{cases} \tag{8}$$

The value of $\alpha$ is a certain bifurcation parameter which switches the stable and unstable states of the system. At lower $\alpha$, $\alpha<\alpha_0=(2/\pi)|\arg(\lambda_i)|$, the system has oscillatory modes, but they are stable. Increasing the value of $\alpha>\alpha_0 =(2/\pi)|\arg(\lambda_i)|$ leads to instability. In fact, having a complex number $\lambda_i$ with $\mathrm{Re}\,\lambda_i<0$, $\mathrm{Im}\,\lambda_i\neq 0$, it is always possible to satisfy the condition $|\arg(\lambda_i)|<\alpha\pi/2$, and the system becomes unstable according to a certain type of oscillation. The smaller the value of $\mathrm{tr}F$, the easier it is to satisfy the instability conditions. In the case of $k=0$ this type of analysis was made in [15,16]. In this paper we would like to direct your attention to the fact that we can have another type of instability when $k\neq 0$, which leads to nonlinear inhomogeneous oscillations of system parameters.

## III. DIFFERENT LIMITS OF INSTABILITY

Let us consider stability conditions for different possible limits. It is widely known for integer time derivatives that the system (1) and (2) becomes unstable according to either a Turing or a Hopf bifurcation.

Conditions for the Turing instability are

$$\mathrm{tr}F < 0, \quad \det F(k=0) > 0, \quad \det F(k_0) < 0. \tag{9}$$

In this case, the eigenvalues are real, and at $a_{11}>0$, $a_{22}<0$, $a_{12}a_{21}<0$, $l \ll L$ the conditions of the Turing instability for $k_0\neq 0$ lead to spatial pattern formation.

Conditions for the Hopf bifurcation are

$$\mathrm{tr}F > 0, \quad \det F(k=0) > 0, \tag{10}$$

which occur at $k=0$, $a_{11}>0$, $a_{22}<0$, $a_{12}a_{21}<0$, $\tau_1<\tau_2$ and lead to homogeneous oscillations.

In the case of a fractional derivative index, the Hopf bifurcation is not connected with the condition $a_{11}>0$ and can hold at a certain value of $\alpha$ when the fractional derivative index is sufficiently large [16].

Let us consider a new possible situation when

$$\mathrm{tr}F < 0, \quad 4 \det F(0) < \mathrm{tr}^2 F(0), \quad 4 \det F(k_0) > \mathrm{tr}^2 F(k_0). \tag{11}$$

Analysis of the expressions (11) shows that at $k=0$ we have two real and less than zero eigenvalues, and the system is certainly stable. If the last inequality occurs for certain values of $k_0\neq 0$, we can get two complex eigenvalues. As a result, in the case of fractional derivatives, a different type of instability, connected with the interplay between the determinant and trace of the linear system, emerges. With this type of eigenvalue, it is possible to find the value of the fractional derivative index when the system becomes unstable.

In fact, the last two conditions can be rewritten as

$$(a_{11}\tau_1 - a_{22}\tau_2)^2 > -4a_{12}a_{21}\tau_1\tau_2, \tag{12}$$

$$-4a_{12}a_{21}\tau_1\tau_2 > [(a_{11}-k^2l^2)\tau_2 - (a_{22}-k^2L^2)\tau_1]^2. \tag{13}$$

The simplest way to satisfy the last condition, is to estimate the optimal value of $k=k_0$,

$$k_0^2 = \left| \frac{a_{11}\tau_2 - a_{22}\tau_1}{\tau_1 L^2 - \tau_2 l^2} \right|. \tag{14}$$

Having obtained (14), we can estimate the marginal value of $\alpha_0$

$$\alpha_0 = 2 - \frac{2}{\pi} \arctan T, \tag{15}$$

where the expression $T$ is

$$T = \frac{(-4a_{12}a_{21}\tau_1\tau_2)^{1/2}}{\left| (a_{11}\tau_2 - a_{22}\tau_1)\dfrac{l^2\tau_2 + L^2\tau_1}{l^2\tau_2 - L^2\tau_1} \right| - a_{11}\tau_2 - a_{22}\tau_1}.$$

The last expression determines the value of $\alpha_0$ as a function of all parameters of the system. The greater the expression, the smaller is the value $\alpha_0$. Trying to reach the maximum possible value of (15), we can see that it goes to zero if either $\tau_1$ or $\tau_2$ goes to zero and, as a result, $\alpha_0 \to 2$. In the intermediate situation, when $\tau_1 \sim \tau_2$ the expression reaches its maximum. Analyzing the last expression, we can see that at $L \sim l$ the denominator is very large and the right-hand side tends to zero. For different lengths $L \ll l$ or $L \gg l$, $\tau_1 \sim \tau_2$, $a_{11}<0$, $a_{22}<0$, $a_{12}a_{21}<0$, we can estimate the value of $\alpha_0$ as $\alpha_0 = 2-(2/\pi)\arctan 1/2 \approx 1.7$.

## IV. COMPUTER SIMULATION OF THE STABILITY CURVES AND INHOMOGENEOUS OSCILLATORY STRUCTURES

We consider here a Bonhoeffer–van der Pol type of RDS with cubic nonlinearity (see, for example, [3,4,21,22]). In this case, the source term for the activator variable is nonlin-

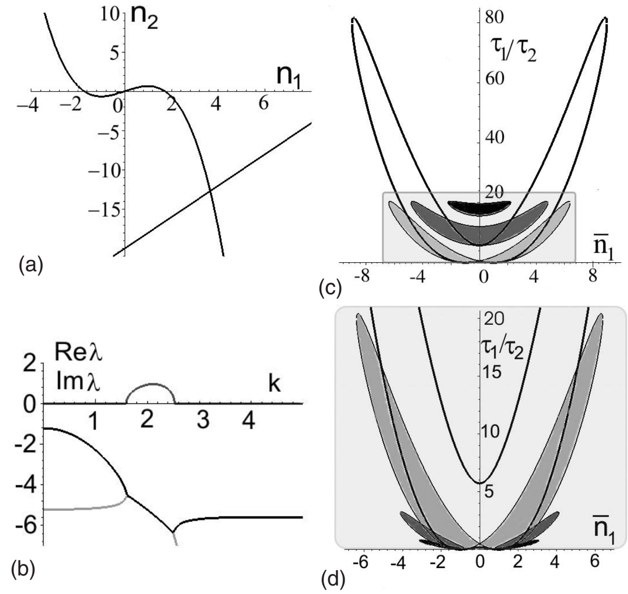


FIG. 1. (a) Null isoclines. (b) Imaginary and real eigenvalues. (c) Two-dimensional bifurcation diagram domain in coordinates $(n_1, \tau_1/\tau_2)$ for $\alpha=1.9$, $\beta=2$, $L=1$, and $k=1$, for different values of $l$ ($l=6$, black; $l=4$, dark gray; $l=0.1$, light gray). (d) Zoomed part of the region (c) at $l=0.1$ for different values of $k$ ($k=3$, black; $k=2$, dark gray; $k=1$, light gray).

ear $W=n_1-n_1^3-n_2$, and it is linear for the inhibitor one $Q=-n_2+\beta n_1+\mathcal{A}$. The null isoclines of the system are represented on Fig. 1(a). The homogeneous solution of variables $\bar{n}_1$ and $\bar{n}_2$ can be determined from the system of equations $W=Q=0$, and, for example, for determination of $\bar{n}_1$ we have the cubic algebraic equation

$$(\beta-1)\bar{n}_1+\bar{n}_1^3/3+\mathcal{A}=0. \tag{16}$$

Simple calculation of the derivatives in the homogeneous state (16) $a_{11}=(1-\bar{n}_1^2)$, $a_{12}=-1$, $a_{21}=\beta$, $a_{22}=-1$ makes it possible to investigate the eigenvalues of the system. Real and imaginary parts for this case are represented on Fig. 1(b). We see that the real part of the roots is always less than zero and the imaginary part in some interval of wave number $k$ becomes nonzero. In this case when the fractional derivative index becomes greater than some critical value $\alpha_0$, the instability condition holds true. So, as these instability conditions are possible to realize for some interval $k_{\min}<k<k_{\max}$, this means that only the perturbations with this wave number are unstable, and they are unstable for oscillatory fluctuations. This situation is qualitatively different from the integer RDS whether either a Turing ($k\neq 0$) or a Hopf bifurcation ($k=0$) takes place, and this depends on which conditions are easier to realize. In the system under consideration, we can choose the parameter when we have no Turing on Hopf bifurcation (for $k=0$) at all. Nevertheless, we obtain the result that conditions for Hopf bifurcation can be realized for nonhomogeneous wave numbers.

Taking into account the calculations made above we can estimate the value of $T$:

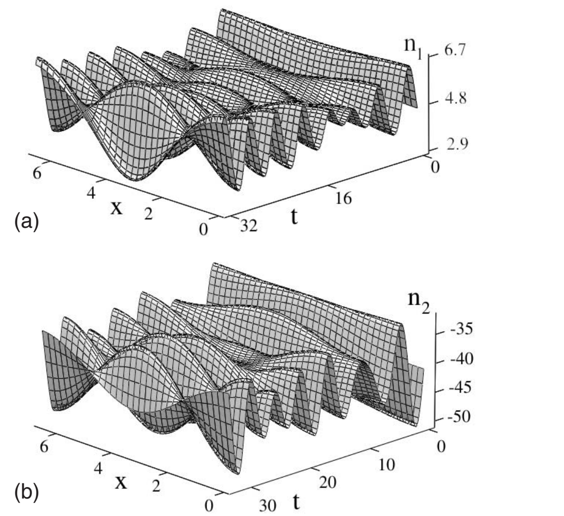


FIG. 2. Oscillatory structures of fractional RDS (1) and (2). Dynamics of variable $n_1$ (a) and $n_2$ (b) on the time interval (0,32) for $\alpha=1.94$, $l_x=6.28$, $\mathcal{A}=-50.0$, $\beta=2$, $\tau_1=12.0$, $\tau_2=1$, $l^2=0.1$, $L^2=1$. Initial conditions are $n_1^0=\bar{n}_1-0.05\cos(k_0 x)$, $n_2^0=\bar{n}_2-0.05\cos(k_0 x)$.

$$T=\frac{2\sqrt{\beta\zeta}}{\left|[(1-\bar{n}_1^2)\zeta+1]\dfrac{l^2\zeta+L^2}{L^2-l^2\zeta}\right|+(\bar{n}_1^2-1)\zeta+1},$$

where $\tau_2/\tau_1=\zeta$, which determines the marginal value of $\alpha_0$ (15).

In Figs. 1(c) and 1(d), two-dimensional plots display the parameter ranges for the stability and existence of dynamical structures. In the largest "boomerang" domain the system is unstable according to wave numbers $k=0$ [15,16]. For the case $k\neq 0$, we find instability conditions for different wave numbers $k=1,2,3,\ldots$ [by solution of the equality $|\arg(\lambda_i)|=\alpha\pi/2$ at certain $\alpha$ (7)]. We can see that these regions overlap and, at the same parameters, the instability conditions for different regimes are satisfied simultaneously [Figs. 1(c) and 1(d)]. As is seen from the figures, there are conditions where only instability according to nonhomogeneous wave numbers holds. As a result, perturbations with $k=0$ relax to the homogenous state; only the perturbations with a certain value of $k$ become unstable and the system exhibits inhomogeneous oscillations.

The results of the numerical study of the initial value problem of the system (1) and (2) are represented on Figs. 2(a) and 2(b). The system with the corresponding initial and boundary conditions was integrated numerically using explicit and implicit schemes with respect to time and the centered difference approximation for spatial derivatives. The fractional derivatives were approximated using a scheme on the basis of the Grunwald-Letnikov definition [19].

In contrast to a standard RDS, here, the inhomogeneous distributions are unstable at certain wave numbers and lead to space-time oscillations. With increase in the parameter $\alpha$, the amplitude of the oscillatory structures increases. The emergence of inhomogeneous oscillations, which destroy the stationary state, leads to a different form of pattern formation. The resulting structures are similar to standing waves, rather than to standard structures already investigated in autowave media.

## V. CONCLUSION

In this Rapid Communication we consider a mechanism of instability in RDSs with fractional derivatives. It was shown that, at a sufficient value of the fractional derivative index $\alpha$, the system becomes unstable to inhomogeneous perturbations ($k \neq 0$) with eigenvalues with imaginary parts. As a result of this instability, pattern formation can be represented as oscillatory structures similar to inhomogeneous standing waves in linear systems.

It should be noted that at the present time we do not have a reliable experimental system for investigation of these phenomena. Nevertheless, systems with anomalous diffusion properties that are described by reaction-diffusion equations can be created synthetically, with the help of modern technology [23–25]. In this case, the corresponding layers have to be endowed with the properties inherent to fractional order controllers [26]. As a result, each layer can be described by fractional differential equations and can even have its own fractional index.

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